Poissan's and Laplace Equation

* Poissan's and laplace equation

From maxwell's eqn $\rho_v = \nabla \cdot \vec{D}$ — (1)

but $\vec{D} = \epsilon \vec{E}$ — (2).

we also know,

$\vec{E} = -\nabla V$ — (3).

∇V is called potential gradient.

using (2) & (3) in (1).

$$\rho_v = \nabla \cdot (\epsilon \vec{E})$$
$$= \nabla \cdot (\epsilon(-\nabla V))$$
$$= -\epsilon(\nabla \cdot \nabla V)$$
$$\rho_v = -\epsilon(\nabla^2 V)$$

$$\Rightarrow \boxed{\nabla^2 V = \frac{-\rho_v}{\epsilon}}$$ — (4). This is called Poissan's equation where charges exist in free space.

If the region is charge free, then $\rho_v = 0$.

eq (4) becomes $\boxed{\nabla^2 V = 0}$ — (5) This is called laplace equation.

Laplacian is a scalar quantity obtained by taking divergence of the operator ∇ with the gradient of V,

$\nabla \cdot (\nabla V)$ where ∇ is del, ∇V is gradient & it is defined as $\nabla^2 V = \nabla \cdot (\nabla V)$.

In Cartesian,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

In Cylindrical,

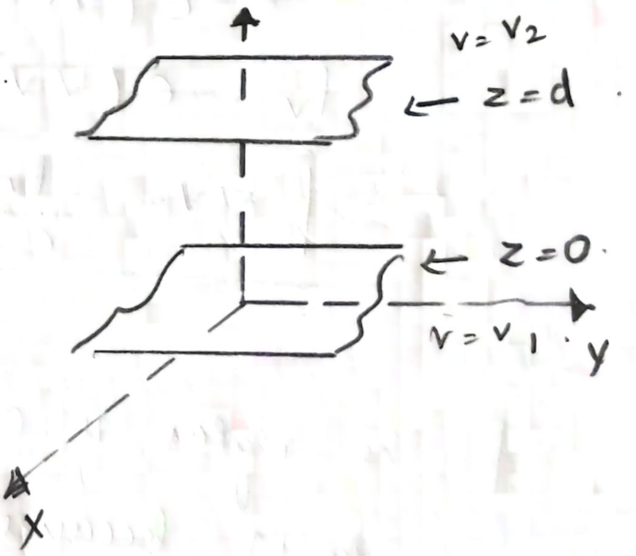
$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\left[\frac{\partial^2 V}{\partial \phi^2}\right] + \frac{\partial^2 V}{\partial z^2} = 0.$$

In Spherical,

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

Applications of Laplace Equation

1. Two uniformly charged parallel planes of infinite extent (case of parallel plate capacitor).



Consider two uniformly charged planes M and N of infinite extent extending in parallel x-y planes and separated by a distance 'd' along the 'z' direction as in the fig.

Plane M is located at $z = 0$ with potential V_1

Plane N is located at $z = d$ with potential V_2.

Region between two planes is free of charges $\therefore \rho_v = 0$.

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad —(1)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{\partial^2 V}{\partial y^2} = 0. \quad —(2)$$

using (2) in (1) $\quad \frac{\partial^2 V}{\partial z^2} = 0. \quad —(3)$

Integrate, $\frac{\partial V}{\partial z} = \int 0\, dz + C_1 = C_1$ ②

Integrate, $V = \int C_1 dz + C_2$

$$\boxed{V = C_1 z + C_2} \quad - ④$$

~~At $z=0$, $V=0$~~

~~$0 = C_1(0) + C_2$~~

$\Rightarrow \boxed{\cancel{C_2 = 0}}$ $-(5)$

At $z=0$, $V = V_1$

$V_1 = C_1(0) + C_2$

$$\boxed{C_2 = V_1} \quad -(5)$$

At $z=d$, $V = V_2$

$V_2 = C_1(d) + C_2$

$$C_1 = \frac{V_2 - C_2}{d} \quad -(6)$$

using ⑤ in ⑥.

$$\boxed{C_1 = \frac{V_2 - V_1}{d}} \quad - ⑦$$

use (5) & (7) in ④.

$$\boxed{V = \left(\frac{V_2 - V_1}{d}\right) z + V_1} \quad - ⑧$$

$$\vec{E} = -\nabla V = -\left[\frac{\partial}{\partial x}\hat{a}_x + \frac{\partial}{\partial y}\hat{a}_y + \frac{\partial}{\partial z}\hat{a}_z\right] V$$

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0$$

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{a}_z \quad -(9)$$

From eq (8)

$$\frac{\partial V}{\partial z} = \frac{V_2 - V_1}{d} \quad \text{— (10)}$$

use (10) in (9).

$$\boxed{\vec{E} = -\frac{(V_2 - V_1)}{d}\hat{a}_z = \frac{V_1 - V_2}{d}\hat{a}_z}$$

$$\vec{E} = -\frac{V_0 \hat{a}_z}{d}, \quad \vec{D} = \epsilon\vec{E} = -\frac{V_0 \epsilon}{d}\hat{a}_z \ c/m^2.$$

$$\rho_s = |\vec{D}_N| = \frac{V_0 \epsilon}{d} \ c/m^2.$$

$Q = \rho_s \times A$; A = Cross sectional area of each plate

$V = V_0$ = potential difference between the plates.

$$C = \frac{Q}{V} = \frac{\rho_s \times A}{V} = \frac{V_0 \epsilon \times A}{d\, V_0} = \frac{\epsilon A}{d} \ F.$$

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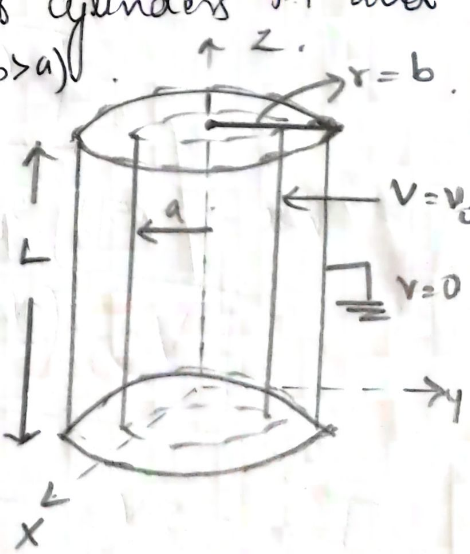
2. Two concentric cylinders of infinite length co-axial Cable :-



→ Considering two conducting right circular cylinders 'M' and 'N' of radii 'a' and 'b' respectively (b>a).

→ Let inner cylinder be uniformly charged and outer cylinder be earthed.

$$\rightarrow \nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

→ potential remains constant for ϕ and z and varies along 'r' co-ordinate.

ie $$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0 \quad \text{— (1)} \Rightarrow \frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0.$$

Integrating on both sides

$$\frac{\partial}{\partial r}\left[r\frac{\partial v}{\partial r}\right]=0.$$

$$r\frac{\partial v}{\partial r} = A. \quad ; \quad A = \text{Constant}.$$

$$\frac{\partial v}{\partial r} = \frac{A}{r}.$$

Integrate again

$$\boxed{V = A\ln r + B} \quad -(1); \quad B = \text{Constant}.$$

Apply boundary condition to evaluate A & B.

Let $V=V_0$ at $r=a$.

and $V=0$ at $r=b$.

using these boundary condition in eq (1).

$$V_0 = A\ln a + B. \quad -(2)$$

$$0 = A\ln b + B.$$

$$\Rightarrow B = -A\ln b. \quad -(3)$$

using (3) in (2)

$$V_0 = A\ln a - A\ln b.$$

$$= A(\ln a - \ln b)$$

$$V_0 = A\ln(a/b) \Rightarrow \boxed{A = \frac{V_0}{\ln(a/b)}} \quad -(4)$$

using (4) in (3).

$$B = -\frac{V_0}{\ln(a/b)}\ln b. \quad -(5)$$

using (4) & (5) in (1).

$$V = \frac{V_0}{\ln(a/b)}\ln r - \frac{V_0}{\ln(a/b)}\ln b = \frac{V_0}{\ln(a/b)}\left[\ln r - \ln b\right]$$

$$\boxed{V = \frac{V_0}{\ln(a/b)} \ln(r/b)} \quad — (6).$$

$$\vec{E} = -\nabla V.$$

$$= -\left[\frac{\partial}{\partial r} V\right] \hat{a}_r$$

$$= -\left[\frac{\partial}{\partial r}\left[\frac{V_0}{\ln(a/b)} \ln(r/b)\right]\right] \hat{a}_r$$

$$= \frac{-V_0}{\ln(a/b)} \frac{\partial}{\partial r}\left(\ln(r/b)\right] \hat{a}_r$$

$$= \frac{-V_0}{\ln(a/b)} \frac{\partial}{\partial r}\left[\ln r - \ln b\right] \hat{a}_r$$

$$= \frac{-V_0}{\ln(a/b)} \left(\frac{1}{r}\right) \hat{a}_r$$

$$\boxed{\vec{E} = \frac{V_0}{r \ln(b/a)} \hat{a}_r} \quad — (7)$$

$$\vec{D} = \epsilon \vec{E} \Rightarrow \vec{E} = \frac{\vec{D}}{\epsilon}.$$

w.k.T $\quad \vec{D} = \frac{\rho}{S} = \frac{Q}{S} = \frac{Q}{2\pi a L} \hat{a}_r$.

$$\boxed{\vec{E} = \frac{Q}{2\pi\epsilon a L} \hat{a}_r} \quad — (8).$$

equating (7) & (8).

$$\frac{V_0}{r \ln(b/a)} \hat{a}_r = \frac{Q}{2\pi\epsilon a L} \hat{a}_r.$$

$$\frac{Q}{V_0} = \frac{2\pi\epsilon a L}{r \ln(b/a)}.$$

[$r = a$ = inner cylinder radius

$$\frac{Q}{V_0} = \frac{2\pi\epsilon a L}{a\ln(b/a)}.$$

$$C = \frac{Q}{V_0} = \frac{2\pi\epsilon a L}{a\ln(b/a)} \quad (F)$$

$$\boxed{C = \frac{2\pi\epsilon L}{\ln(b/a)}} \quad F.$$

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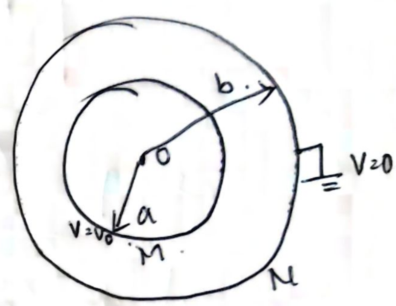
3. Two concentric spheres.

→ Consider two concentric spherical shells M and N of radii a & b respectively (b > a).



→ $$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} = 0.$$

potential remains constant for θ and ϕ, so.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0.$$

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0. \quad —(1)$$

Integrating,

$$r^2\frac{\partial V}{\partial r} = A.$$

$$\frac{\partial V}{\partial r} = \frac{A}{r^2}.$$

Integrating again.

$$\boxed{V = -\frac{A}{r} + B} \quad —(2)$$

Boundary Conditions,

$V = V_0$ at $r = a$

$V = 0$ at $r = b$.

using B.Cs in eq (2).

$$0 = -\frac{A}{b} + B.$$

$$\boxed{B = \frac{A}{b}} \quad -(3)$$

$$V_0 = -\frac{A}{a} + B.$$

$$V_0 = -\frac{A}{a} + \frac{A}{b} \Rightarrow V_0 = A\left(1/b - 1/a\right).$$

$$\boxed{A = \frac{V_0}{(1/b - 1/a)}} \quad -(4).$$

using (4) in (3).

$$\boxed{B = \frac{V_0}{b\,(1/b - 1/a)}} \quad -(5).$$

using (4) & (5) in (2).

$$V = \frac{-V_0}{r\,(1/b - 1/a)} + \frac{V_0}{b(1/b - 1/a)}.$$

$$V = \frac{V_0}{(1/b - 1/a)}\left[1/b - 1/r\right].$$

$$\boxed{V = \frac{V_0\,(1/r - 1/b)}{(1/a - 1/b)}} \quad -(6).$$

Since the potential undergoes variation only in radial direction and is independent of ϕ & z.

$$\vec{E} = -\left(\frac{d}{dr} V\right)\hat{a}_r$$

$$= -\frac{\partial}{\partial r}\left[\frac{V_0\,(1/r - 1/b)}{(1/a - 1/b)}\right]\hat{a}_r$$

$$= \frac{-V_0}{(1/a - 1/b)}\frac{\partial}{\partial r}\left(1/r - 1/b\right)\hat{a}_r = \frac{-V_0}{(1/a - 1/b)}\left(-1/r^2\right)\hat{a}_r.$$

$$\boxed{\vec{E} = \frac{V_0}{r^2(1/a - 1/b)}\hat{a}_r} \quad \text{— (7)}$$

$$\vec{D} = \frac{Q}{S} = \frac{Q}{4\pi r^2}\hat{a}_r$$

$$\vec{D} = \epsilon\vec{E} \Rightarrow \vec{E} = \frac{\vec{D}}{\epsilon} = \frac{Q}{4\pi r^2\epsilon}\hat{a}_r \quad \text{— (8)}$$

equaling (7) & (8).

$$\frac{V_0}{r^2(1/a - 1/b)}\hat{a}_r = \frac{Q}{4\pi\epsilon r^2}\hat{a}_r$$

$$\frac{Q}{V_0} = \cancel{1/a - 1/b}$$

$$\boxed{C = \frac{Q}{V_0} = \frac{4\pi\epsilon}{(1/a - 1/b)}\ F}$$

~~Continuity Equation~~

Uniqueness Theorem :- This theorem exhibits the uniqueness to the solution of laplace equation.

"Under the given boundary conditions, laplace equation has one and only one solution".

W.r.T $\nabla^2 V = 0$ (Laplace equation)

Let V_1 and V_2 be the two solutions that obeys the above equation and satisfies the boundary conditions for the given case.

$$\nabla^2 V_1 = 0 \quad \text{— (1)}$$

$$\nabla^2 V_2 = 0 \quad \text{— (2)}$$

(1) − (2), $\nabla^2(V_1 - V_2) = 0$. $\quad f = V_1 - V_2$.

$$\nabla^2 f = 0 \quad \text{— (3)}$$

Let the potential at the boundary be V_b.

Let the values of the solutions at the boundary be V_{1b} and V_{2b} respectively & $f = f_b$.

$\Rightarrow$ $f - f_b = V_{1b} - V_{2b} = 0$. — 4 .

If $\vec{A}$ is any vector and f as a scalar function must obey

$$\nabla \cdot (f\vec{A}) = f(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla f) \quad — 5$$

∇f is a vector

let $\vec{A} = \nabla f$.

Then eqn 5 becomes.

$$\nabla \cdot f(\nabla f) = f[\nabla \cdot (\nabla f)] + (\nabla f) \cdot (\nabla f) \quad — 6$$

Integrating the above equation over the entire volume V enclosed by the surface defined by boundary condition, then

$$\int_V \nabla \cdot f(\nabla f)\, dv = \int_V f[\nabla \cdot (\nabla f)]\, dv + \int_V (\nabla f)^2 dv . \quad — 7$$

we have from divergence theorem.

$$\int_V \nabla \cdot \vec{A}\, dv = \oint_S \vec{A} \cdot \vec{ds}$$

$$\text{or} \quad \int_V \nabla \cdot (\nabla f)\, dv = \oint_S \nabla f \cdot \vec{ds} \quad — 8$$

multiply both sides by f

$$\int_V (\nabla \cdot f)(\nabla f)\, dv = \oint_S f \cdot \nabla f \cdot \vec{ds} \quad — 9$$

The right side of the above equation is an integral over a closed surface that surrounds the volume. This surface involves the boundaries at which as we know $f = f_b = 0$.

∴ RHS of eq 9 is zero.

L.H.S of eq 9 is also zero.

L.H.S of eq 8 is zero.

Now, as per vector algebra

$$\nabla \cdot (\nabla f) = \nabla^2 f .$$

Since $\nabla^2 f = 0$.

$\therefore$ first term in the R.H.S of eq (9) is zero.

Thus,

$$\int_V (\nabla f)^2 dv = 0.$$

$$\int_V [\nabla (V_1 - V_2)]^2 dv = 0 \quad \text{— (10)}$$

An integral over a certain volume can be zero under two conditions:

i) If the integrand has the values at some parts of the volume and -ve at some other parts of the volume so that the sum could be zero.

ii) If the integrand is zero at every point in ~~the~~ volume.

In eq (10) since the integrand consists of a square term it cannot be -ve at any point which rules out the first possibility.

hence,

$$\nabla (V_1 - V_2)^2 = 0.$$

$$(V_1 - V_2)^2 = a = \text{Constant}$$

ie $V_1 - V_2 = a$ = Constant over the entire volume.

It implies that the value it has at the boundary and that inside the volume should be equal.

$$\therefore V_1 - V_2 = V_{1b} - V_{2b} = 0$$

$$\therefore \boxed{V_1 = V_2}.$$

Thus the two solutions are identical & hence the uniqueness theorem.

(1) Verify the potential field given below satisfied the laplace equation a) $V = 2x^2 - 3y^2 + z^2$. (1)

:- given $V = 2x^2 - 3y^2 + z^2$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

$$\frac{\partial V}{\partial x} = 4x \;;\; \frac{\partial^2 V}{\partial x^2} = 4$$

$$\frac{\partial V}{\partial y} = -6y \;;\; \frac{\partial^2 V}{\partial y^2} = -6.$$

$$\frac{\partial V}{\partial z} = 2z \;;\; \frac{\partial^2 V}{\partial z^2} = 2.$$

$$\nabla^2 V = 4 - 6 + 2 = 0.$$

Since $\nabla^2 V = 0$; the given potential field satisfies the laplace equation.

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b) $\vec{E} = 5\cos z\,\hat{a}_z$

:- $\vec{E} = -\nabla V$.

$$-\nabla V = 5\cos z\,\hat{a}_z$$

$$\nabla\cdot\nabla V = -\nabla\cdot(5\cos z\,\hat{a}_z)$$

$$\nabla^2 V = -\left[\frac{\partial \hat{a}_z}{\partial z}(5\cos z\,\hat{a}_z)\right].$$

$$= -\frac{\partial(5\cos z)}{\partial z}$$

$$\boxed{\nabla^2 V = 5\sin z \neq 0}$$ hence the given region is not free of charges

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c) $V = r\cos\phi + z$.

:- cylindrical co-ordinate system.

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2}. \quad -(1)$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}[r\cos\phi + z) = \cancel{\quad} \cos\phi$$

$$r\frac{\partial V}{\partial r} = + \cdot r\cos\phi$$

$$\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = \cos\phi$$

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = \frac{1}{r}\cos\phi} \quad -(2)$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}[r\cos\phi + z] = -r\sin\phi$$

$$\frac{\partial^2 V}{\partial \phi^2} = -r\cos\phi$$

$$\boxed{\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = -\frac{1}{r}\cos\phi} \quad -(3)$$

$$\frac{\partial V}{\partial z} = \frac{\partial}{\partial z}[r\cos\phi + z] = 1$$

$$\boxed{\frac{\partial^2 V}{\partial z^2} = 0} \quad - 4$$

use eqn 2 3 & 4 in 1.

$$\nabla^2 V = \frac{1}{r}\cos\phi - \frac{1}{r}\cos\phi + 0$$

$\boxed{\nabla^2 V = 0}$ hence satisfies laplace equation

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d). ~~$V = r\cos\phi$~~ $V = r\cos\theta + \phi$.

Spherical coordinate system

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}.$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left[r\cos\theta + \phi\right] = \cos\theta .$$

$$r^2\frac{\partial V}{\partial r} = r^2\cos\theta$$

$$\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = 2r\cos\theta .$$

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = \frac{2}{r}\cos\theta} \quad — (2)$$

$$\frac{\partial V}{\partial \theta} = \frac{\partial}{\partial \theta}\left[r\cos\theta + \phi\right] = -r\sin\theta .$$

$$\sin\theta\frac{\partial V}{\partial \theta} = -r\sin^2\theta .$$

$$\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] = -2r\sin\theta\cos\theta .$$

$$\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left[\sin\theta\frac{\partial V}{\partial \theta}\right] = \frac{-2r\sin\theta\cos\theta}{r^2\sin\theta} = \frac{-2\cos\theta}{r} \quad — (3)$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}\left[r\cos\theta + \phi\right] = 1 .$$

$$\frac{\partial^2 V}{\partial \phi^2} = 0 .$$

$$\boxed{\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi} = 0 .} \quad — (4)$$

use (2) (3) (4) in (1).

$$\nabla^2 V = \frac{2}{r}\cos\theta - \frac{2}{r}\cos\theta + 0 = 0 .$$ hence satisfies laplace equation.

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(2) Given the potential field $V = \left[Ar^4 + Br^{-4}\right]\sin 4\phi$. S.T $\nabla^2 V = 0$, Select A and B so that $V = 100$ volt & $|E| = 500$ V/m at $P(r=1, \phi = 22.5^\circ, z = 2)$.

$$\therefore \nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} .$$

$$\frac{\partial V}{\partial r} = \frac{\partial}{\partial r}\left[A r^{4} + B r^{-4}\right] \sin 4\phi .$$

$$= \left(4A r^{3} - 4B r^{-5}\right) \sin 4\phi .$$

$$\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) = \frac{\partial}{\partial r}\left[r\left(4A r^{3} - 4B r^{-5}\right) \sin 4\phi\right] .$$

$$= \left(16 A r^{3} + 16 B r^{-5}\right) \sin 4\phi .$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial V}{\partial r}\right) = 16\left(A r^{2} + B r^{-6}\right) \sin 4\phi . \quad \text{— (2)}$$

$$\frac{\partial V}{\partial \phi} = \frac{\partial}{\partial \phi}\left(A r^{4} + B r^{-4}\right) \sin 4\phi .$$

$$= \left(A r^{4} + B r^{-4}\right) 4 \cos 4\phi .$$

$$\frac{\partial^{2} V}{\partial \phi^{2}} = \left(A r^{4} + B r^{-4}\right)\left(-16 \sin 4\phi\right) \quad \text{—)} .$$

$$\frac{1}{r^{2}}\frac{\partial^{2} V}{\partial \phi^{2}} = -16\left(A r^{2} + B r^{-6}\right) \sin 4\phi \quad \text{— (3)} .$$

$$\frac{\partial V}{\partial z} = 0 \Rightarrow \frac{\partial^{2} V}{\partial z^{2}} = 0 \quad \text{— (4)} .$$

using (2) (3) & (4) in (1) .

$$\nabla^{2} V = 16\left(A r^{2} + B r^{-6}\right) \sin 4\phi - 16\left(A r^{2} + B r^{-6}\right) \sin 4\phi + 0$$

$$\boxed{\nabla^{2} V = 0} .$$

$$V = \left[A r^{4} + B r^{-4}\right] \sin 4\phi .$$

$$100 = \left[A(1)^{4} + B(1)^{-4}\right] \sin\left(4 \times 22.5^{\circ}\right) .$$

$$\boxed{100 = A + B .} \quad \text{— (1)}$$

given $|E| = 500 \text{ V/m}$.

$$\nabla V = \left[\frac{\partial V}{\partial r}\hat{a}_r + \frac{1}{r}\frac{\partial V}{\partial \phi}\hat{a}_\phi + \frac{\partial V}{\partial z}\hat{a}_z\right]$$

$$\nabla V = \left[4Ar^3 - 4Br^{-5} \sin 4\phi\right]\hat{a}_r + \left[(Ar^3 + Br^{-5})4\cos 4\phi\right]\hat{a}_\phi$$ (9)

$r = 1, \quad \phi = 22.5^\circ$.

$$\nabla V = (4A - 4B)\hat{a}_r$$

$$\vec{E} = -\nabla V = -(4A - 4B)\hat{a}_r$$

$$|E| = 500 = 4A - 4B.$$

$$\boxed{125 = A - B.} \quad -(2).$$

Solving (1) and (2)

$$\boxed{A = 112.5.} \qquad \boxed{B = -12.5.}$$

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(3) A sphere of radius 'a' has the charge distribution $\rho(v)$ C/m³ which produces an electric field intensity given by,

$$E_r = Ar^4 \quad \text{for } r \le a.$$

$$= Ar^{-2}, \quad \text{for } r > a.$$

where A is a constant. Find the corresponding charge distribution $\rho(v)$.

$$\therefore \quad \nabla^2 V = \frac{-\rho_v}{\epsilon} \quad -(1)$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = -\frac{\rho_v}{\epsilon} \quad \text{(spherical)} \quad -(2).$$

Case i). $r \le a$.

$$E_r = Ar^4.$$

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial r} = Ar^4.$$

$$\frac{\partial V}{\partial r} = -Ar^4$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial V}{\partial r}\right] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 (-Ar^4)\right] = \frac{-A}{r^2} 6r^5.$$

$$\nabla^2 V = -6Ar^3 \quad -(3).$$

from (1) & (3)

$$-\frac{\rho_v}{\epsilon} = -6Ar^3 \Rightarrow \boxed{\rho_v = 6\epsilon A r^3}$$

Case ii) :- $r > a$.

$E_r = A/r^2$.

$$\bar{E} = -\nabla V = -\frac{\partial V}{\partial r} = \frac{A}{r^2}.$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial V}{\partial r}\right] = 0.$$

$$\boxed{\rho = 0}$$

---

4) In a certain region the volume charge density is constant and is equal to $1\,\mu C/m^3$ in the x-direction at
i) $x = 0.1m$, $V = 100V$
ii) $x = 0m$, $V = 0V$. Determine the expression for potential.

:- By poisson's equation

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

Since it is a function of 'x' only we have

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho_v}{\epsilon}.$$

Integrating w.r.t 'x'

$$\frac{\partial V}{\partial x} = -\frac{\rho_v}{\epsilon}x + A.$$

Again integrating

$$\boxed{V = -\frac{\rho_v}{\epsilon}\frac{x^2}{2} + Ax + B} \quad -(1).$$

Boundary Condition,
i) $x = 0.1$, $V = 100V$
ii) $x = 0m$, $V = 0V$.

using these in eq (1)

$$\boxed{B = 0} \quad \text{and} \quad \boxed{100 = -\frac{\rho_v}{\epsilon}\left(\frac{0.1^2}{2}\right) + 0.1A + B}$$

on solving, $A = 6647.16$, $B = 0$.

$$\boxed{V = -56470.4x^2 + 6647.16x}$$

(5) Find the potential at $(2,1,3)$ m for the field of two co-axial cylindrical conductors.

G.T $V = 50V$ at $r = 2m$

$V = 20V$ at $r = 3m$.

:- $\nabla^2 V = 0$.

$r = \sqrt{x^2+y^2} = 2.23$.

$\phi = \tan^{-1}(y/x) = 0.463$ rad.

$z = 3$.

Since potential is a function of 'r' only

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0.$$

$$\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right] = 0.$$

Integrating w.r.t. r

$$r\frac{\partial V}{\partial r} = A.$$

$$\frac{\partial V}{\partial r} = \frac{A}{r}.$$

Integrating,

$$\boxed{V = A\ln r + B}$$

using B.C,

$A = -73.9$, $B = 101.25$.

$V = -73.9\ln r + 101.25$.

$$\boxed{V_{(2,1,3)} = 41.75}$$

(6) G.T $V = C/R$ where 'C' is a constant and R is a variable. Check whether 'V' satisfies Laplace equation.

:- $\nabla^2 V = 0$.

Let us check for all 3 co-ordinate system.

a) Cartesian system

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

here $x = R$, $y = z = 0$.

$$\frac{\partial^2 V}{\partial R^2}, \quad \frac{\partial V}{\partial R} = \frac{-C}{R^2}.$$

$$\frac{\partial^2 V}{\partial R^2} = \frac{2C}{R^3} \neq 0$$ $\therefore$ the given function doesnot satisfy in Cartesian system.

b) Cylindrical Coordinate system

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial V}{\partial r}\right].$$

$$\frac{\partial V}{\partial R} = -C/R^2.$$

$$\frac{\partial}{\partial R}(-C/R) = C/R^2.$$

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial V}{\partial R}\right) = C/R^3 \neq 0.$$ $\therefore$ The expression for potential doesn't satisfy in cylindrical.

c) Spherical co-ordinate system.

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right).$$

$$\frac{\partial V}{\partial R} = \frac{-C}{R^2}, \quad \frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) = 0.$$

$\therefore$ the given function satisfies

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(7) Find the potential at (2, 1, 3) m for the field of a radial conducting planes with

$V = 50V$ at $\phi = 10^\circ$.

$V = 20V$ at $\phi = 30^\circ$.

Also find the E.F.I and E.F.D at the same point.

:- Assuming Cylindrical system and $P_v = 0$.

$$\frac{\partial^2 v}{\partial \phi^2} = 0.$$

$$\boxed{V = A\phi + B.}$$

$$50 = A \times 10 \times \frac{\pi}{180} + B.$$

$$20 = A \times 30 \times \frac{\pi}{180} + B.$$

on solving

$A = -86.8$, $B = 65$.

$$\overline{E} = -\nabla V.$$

$$= -\frac{1}{r}\frac{\partial v}{\partial \phi}\hat{a}_\phi = -\frac{1}{r}(-86.8)\hat{a}_\phi = 38.4.$$

$$\overline{D} = \epsilon \overline{E} = 3.4 \times 10^{-10}\,\hat{a}_\phi$$

$$V = -86.8\,\phi + 64.9.$$

$$= -86.8\,(0.46) + 64.9$$

$$\boxed{V = 24.972.}$$

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(8) Find V at P(2, 1, 3) for the field of two coaxial conducting cones with V = 50 V at $\theta = 30°$ and V = 20 V at $\theta = 50°$.

:- V is a function of θ only

$$\therefore \nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial v}{\partial\theta}\right) = 0.$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v}{\partial\theta}\right) = 0.$$

Integrating $\quad \sin\theta \dfrac{\partial v}{\partial\theta} = A.$

$$\frac{\partial v}{\partial\theta} = \frac{A}{\sin\theta} = A\operatorname{cosec}\theta.$$

Integrating.

$$V = A\ln(\tan\theta/2) + B.$$

using B.C. and solving

$A = -54.152$, $B = -21.3125$.

$V = -54.152 \ln(\tan \theta/2) - 21.3125$.

for $P(2,1,3)$ $\theta = \cos^{-1} \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos^{-1}\left(3/\sqrt{14}\right)$.

$$\boxed{\theta = 36.669^\circ}$$

$$\boxed{V_p = 38.4489 \text{ V.}}$$